

Upper bound for the Dvoretzky dimension in Milman-Schechtman theorem

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Abstract

For a symmetric convex body $K \subset \mathbb{R}^n$, the Dvoretzky dimension $k(K)$ is the largest dimension for which a random central section of K is almost spherical. A Dvoretzky-type theorem proved by V. D. Milman in 1971 provides a lower bound for $k(K)$ in terms of the average $M(K)$ and the maximum $b(K)$ of the norm generated by K over the Euclidean unit sphere. Later, V. D. Milman and G. Schechtman obtained a matching upper bound for $k(K)$ in the case when $\frac{M(K)}{b(K)} > c(\frac{\log(n)}{n})^{\frac{1}{2}}$. In this paper, we will give an elementary proof of the upper bound in Milman-Schechtman theorem which does not require any restriction on $M(K)$ and $b(K)$.

1 Introduction

Given a symmetric convex body K in \mathbb{R}^n , we have a corresponding norm $\|x\|_K = \inf\{r > 0, x \in rK\}$. Let $|\cdot|$ denote the Euclidean norm, ν_n denote the normalized Haar measure on the Euclidean sphere, S^{n-1} , and $\nu_{n,k}$ denote the normalized Haar measure on the Grassmannian manifold $Gr_{n,k}$. Let $M = M(K) := \int_{S^{n-1}} \|x\|_K d\nu_n$ and $b = b(K) := \sup\{\|x\|_K, x \in S^{n-1}\}$ be the mean and the maximum of the norm over the unit sphere.

In 1971, V. D. Milman proved the following Dvoretzky-type theorem [3]:

Theorem 1. *Let K be a symmetric convex body in \mathbb{R}^n . Assume that $\|x\|_K \leq b|x|$ for all $x \in \mathbb{R}^n$. For any $\epsilon \in (0, 1)$, there is $k \geq C_\epsilon(M/b)^2 n$ such that*

$$\nu_{n,k}\{F \in G_{n,k} : (1 - \epsilon)M < \|\cdot\|_{K \cap F} < (1 + \epsilon)M\} > 1 - \exp(-\tilde{c}k)$$

where $\tilde{c} > 0$ is a universal constant, $C_\epsilon > 0$ is a constant only depending on ϵ .

The quantity C_ϵ was of the order $\epsilon^2 \log^{-1}(\frac{1}{\epsilon})$ in the original proof of V. D. Milman. It was improved to the order of ϵ^2 by Y. Gordon [2] and later, with a simpler argument, by G. Schechtman [6].

In 1997, V. D. Milman and G. Schechtman [5] found that the bound on k appearing in Theorem 1 is essentially optimal. More precisely, they proved the following theorem.

Theorem A. (Milman–Schechtman, see e.g., section 5.3 in [1]). Let K be a symmetric convex body in \mathbb{R}^n . For $\epsilon \in (0, 1)$, define $k(K)$ to be the largest dimension k such that

$$\nu_{n,k}(\{F \in G_{n,k} : \forall x \in S^{n-1} \cap F, (1 - \epsilon)M < \|x\|_K < (1 + \epsilon)M\}) > p_{n,k} = \frac{n}{n+k}.$$

Then,

$$\tilde{C}_\epsilon n(M/b)^2 \geq k(K) \geq \bar{C}_\epsilon n(M/b)^2$$

when $\frac{M}{b} > c(\frac{\log(n)}{n})^{\frac{1}{2}}$ for some universal constant c , where $\|\cdot\|_F$ denotes the norm corresponding to the convex body $K \cap F$ in F , and $\tilde{C}_\epsilon, \bar{C}_\epsilon > 0$ are constants depending only on ϵ .

Because the Dvoretzky–Milman theorem cannot guarantee the lower bound with small $\frac{M}{b}$ for $p_{n,k} = \frac{n}{n+k}$, the original proof required an assumption that $\frac{M}{b} > c(\frac{\log(n)}{n})^{\frac{1}{2}}$ for some c . In [1, p. 197], S. Artstein-Avidan, A. A. Giannopoulos, and V. D. Milman addressed it as an open question whether one can prove the same result when $p_{n,k}$ is a constant, such as $\frac{1}{2}$. When $p_{n,k} = \frac{1}{2}$, the lower estimate on $k(K)$ is a direct result of Dvoretzky–Milman theorem [3], but the upper bound was unknown. In this paper, we are going to give upper bound estimate with $p_{n,k} = \frac{1}{2}$, our main result is the following theorem:

Theorem B. Let K be a symmetric convex body in \mathbb{R}^n . Fix a constant $\epsilon \in (0, 1)$, let $k(K)$ be the largest dimension such that

$$\nu_{n,k}\{F \in G_{n,k} : (1 - \epsilon)M < \|\cdot\|_{K \cap F} < (1 + \epsilon)M\} > \frac{1}{2}.$$

Then,

$$Cn(M/b)^2 \geq k(X) \geq \bar{C}_\epsilon n(M/b)^2$$

where $C > 0$ is a universal constant and $\bar{C}_\epsilon > 0$ is a constant depending only on ϵ .

In the next section, we will provide a proof of Theorem B with no restriction on $\frac{M}{b}$. In fact, from the proof, one can see that $\frac{1}{2}$ can be replaced by any $c \in (0, 1)$ or $1 - \exp(-\tilde{c}k)$, which is the probability appearing in Milman–Dvoretzky theorem.

2 Proof of Theorem B

Let P_k be the orthogonal projection from S^{n-1} to some fixed k -dimensional subspace, and $|\cdot|$ be the Euclidean norm. The upper estimate is related to the distribution of $|P_k(x)|$, where x is uniformly distributed on S^{n-1} .

Recall the concentration inequality for Lipschitz functions on the sphere (see, e.g., [4]):

Theorem 2 (Measure Concentration on S^{n-1}). *Let $f : S^{n-1} \rightarrow \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant b . Then, for every $t > 0$,*

$$\nu_n(\{x \in S^{n-1} : |f(x) - \mathbb{E}(f)| \geq bt\}) \leq 4 \exp(-c_0 t^2 n)$$

where $c_0 > 0$ is a universal constant.

Theorem 2 implies the following elementary lemma.

Lemma 3. *Fix any $c_1 > 0$, let P_k be an orthogonal projection from \mathbb{R}^n to some subspace \mathbb{R}^k . If $t > \frac{c_1}{\sqrt{n}}$ and $\nu_n(\{x \in S^{n-1} : |P_k(x)| < t\}) > \frac{1}{2}$, then $k < c_2 t^2 n$, where $c_2 > 0$ is a constant depending only on c_1 .*

Proof. $|P_k(x)|$ is a 1-Lipschitz function on S^{n-1} with $\mathbb{E}|P_k(x)|$ about $\sqrt{\frac{k}{n}}$. If we want the measure of $\{x : |P_k(x)| < t\}$ to be greater than $1/2$, then measure concentration will force $\mathbb{E}|P_k|$ to be bounded by the size of t , which means $k < c_2 t^2 n$ for some universal constant c_2 . Since $t^2 n > c_1^2$, we may and shall assume k is bigger than some absolute constant in our proof, then adjust c_2 .

To make it precise, we will first give a lower bound on $\mathbb{E}|P_k|$. By Theorem 2,

$$\nu_n(|P_k(x)| - \mathbb{E}|P_k(x)| > t) \leq 4 \exp(-c_0 t n).$$

Thus,

$$\begin{aligned} \mathbb{E}|P_k|^2 - (\mathbb{E}|P_k|)^2 &= \mathbb{E}(|P_k|(x) - \mathbb{E}|P_k|)^2 \\ &< \int_0^\infty \nu_n(|P_k(x)| - \mathbb{E}|P_k(x)| > t) dt \\ &\leq \int_0^\infty 4 \exp(-c_0 t n) dt = \frac{4}{c_0 n}. \end{aligned}$$

With $\mathbb{E}|P_k|^2 = \mathbb{E} \sum_{i=1}^k |x_i|^2 = \frac{k}{n}$, we get $\mathbb{E}(|P_k|) > \sqrt{\frac{k}{n} - \frac{4}{c_0 n}}$. If we assume that $k > \frac{24}{c_0}$, then we have

$$\mathbb{E}(|P_k|) > \sqrt{\frac{1}{2} \frac{k}{n}}.$$

Assuming $k > 8t^2 n$, we have

$$\mathbb{E}(|P_k|) - t > \sqrt{\frac{1}{2} \frac{k}{n}} - t \geq \frac{1}{2} \sqrt{\frac{1}{2} \frac{k}{n}} > 0.$$

Applying Theorem 2 again, we obtain

$$\begin{aligned} \nu_n(|P_k| < t) &< \nu_n(|P_k| - \mathbb{E}|P_k| > \mathbb{E}(|P_k|) - t) \leq 4 \exp(-c_0 (\mathbb{E}(|P_k|) - t)^2 n) \\ &\leq 4 \exp(-c_0 (\frac{1}{2} \sqrt{\frac{1}{2} \frac{k}{n}})^2 n) \leq 4 \exp(-\frac{c_0}{8} k) \leq 4 \exp(-3) < \frac{1}{2}, \end{aligned}$$

which proves our result by contradiction. \square

Theorem 4. Let K be a convex body with inradius $\frac{1}{b}$. For $\epsilon \in (0, 1)$, let k be the largest integer such that

$$\nu_{n,k}\{F \in G_{n,k} : (1 - \epsilon)M < \|\cdot\|_{K \cap F} < (1 + \epsilon)M\} > \frac{1}{2}.$$

Then $k < Cn(\frac{M}{b})^2$ where C is an absolute constant.

Proof. We may assume $\|e_1\|_K = b$, then $K \subset S = \{x \in \mathbb{R}^n : |x_1| < \frac{1}{b}\}$, thus $\|x\|_K \geq \|x\|_S = b|\langle x, e_1 \rangle|$. This implies

$$\begin{aligned} & \{V \in G_{n,k} : \forall x \in V \cap S^{n-1}, (1 - \epsilon)M < \|x\|_K < (1 + \epsilon)M\} \\ \subset & \{V \in G_{n,k} : \forall x \in V \cap S^{n-1}, \|x\|_S < (1 + \epsilon)M\} \\ = & \{V \in G_{n,k} : \sup_{x \in V \cap S^{n-1}} \langle x, e_1 \rangle < (1 + \epsilon)\frac{M}{b}\} \\ = & \{V \in G_{n,k} : |P_V(e_1)| < (1 + \epsilon)\frac{M}{b}\} \end{aligned} \tag{1}$$

where P_V is the orthogonal projection from \mathbb{R}^n to V . If V is uniformly distributed on $G_{n,k}$ and x is uniformly distributed on S^{n-1} , then $|P_{V_0}(x)|$ and $|P_V(e_1)|$ are equi-distributed for any fixed k -dimensional subspace V_0 . Therefore,

$$\nu_{n,k}(\{V \in G_{n,k} : |P_V(e_1)| < (1 + \epsilon)\frac{M}{b}\}) = \nu_n(\{x \in S^{n-1} : |P_{V_0}(x)| < (1 + \epsilon)\frac{M}{b}\}).$$

As shown in the Remark 5.2.2(iii) of [1, p. 164], the ratio $\frac{M}{b}$ has a lower bound $\frac{c'}{\sqrt{n}}$. Setting $c_1 = c'$ and $t = (1 + \epsilon)\frac{M}{b}$, it is easy to see that if

$$\nu_{n,k}\{F \in G_{n,k} : (1 - \epsilon)M < \|\cdot\|_{K \cap F} < (1 + \epsilon)M\} > \frac{1}{2},$$

then $k \leq c_1(1 + \epsilon)^2 \left(\frac{M}{b}\right)^2 n < Cn(\frac{M}{b})^2$ by Lemma 3 and (1). □

Now we can prove Theorem B as a corollary of Theorem 4 and Theorem 1:

Proof of Theorem B. Theorem 1 shows that if $C_\epsilon(M/b)^2 n > \frac{\log(2)}{\tilde{c}}$, then there is $k \geq C_\epsilon(M/b)^2 n$ such that

$$\nu_{n,k}\{F \in G_{n,k} : (1 - \epsilon)M < \|\cdot\|_F < (1 + \epsilon)M\} > 1 - \exp(-\tilde{c}k) > \frac{1}{2}.$$

Otherwise, $(M/b)^2 n < \frac{\log(2)}{\tilde{c}C_\epsilon}$. Therefore, $k(K) \geq \min\{\frac{\tilde{c}C_\epsilon}{\log(2)}, C_\epsilon\}(M/b)^2 n$. Combining it with Theorem 4, we get

$$C\left(\frac{M}{b}\right)^2 n \geq k(K) \geq \min\left\{\frac{\tilde{c}C_\epsilon}{\log(2)}, C_\epsilon\right\}(M/b)^2 n.$$

□

Remark. (1) It is worth noticing that the number $\frac{1}{2}$ plays no special role in our proof. Thus, if we define the Dvoretzky dimension to be the largest dimension such that

$$\nu_{n,k}\{F \in G_{n,k} : (1 - \varepsilon)M < \|\cdot\|_{K \cap F} < (1 + \varepsilon)M\} > c$$

for some $c \in (0, 1)$, then exactly the same proof will work. We will still have $k(K) \sim (\frac{M}{b})^2 n$. Similarly, if we fix ϵ and replace $\frac{1}{2}$ by $1 - \exp(-\tilde{c}k)$, then the lower bound of $k(K)$ is the one from Theorem 1. For k bigger than some absolute constant, we have $1 - \exp(-\tilde{c}k) > \frac{1}{2}$. Thus, the upper bound is still of order $(\frac{M}{b})^2 n$. Therefore, we can replace $\frac{1}{2}$ by $1 - \exp(-\tilde{c}k)$ in Theorem A. With this probability choice, it also shows Theorem 1 provides an optimal k depending on M, b .

(2) Usually, we are only interested in $\epsilon \in (0, 1)$. In the lower bound, $\bar{C}_\epsilon = o_\epsilon(1)$. It is a natural question to ask if we could improve the upper bound from a universal constant C to $o_\epsilon(1)$. Unfortunately, it is not possible due to the following observation. Let $K = \text{conv}(B_2^n, Re_1)^\circ$. By passing from the intersection on K to the projection of K° , one can show that $k(K)$ does not exceed the maximum dimension k such that $\nu_n(P_k(Rx) < 1 + \epsilon) > \frac{1}{2}$. Choosing $R = \sqrt{\frac{n}{l}}$, we get $n(\frac{M}{b})^2 \sim l$ and $k(X) \sim l$ by Theorem 2 and a similar argument to that of Lemma 3. This example shows that no matter what $\frac{M}{b}$ is, one can not improve the upper bound in Theorem A from an absolute constant C to $o_\epsilon(1)$.

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References

- [1] S. Artstein-Avidan, A. A. Giannopoulos, and V. D. Milman, Asymptotic geometric analysis. Part I. Mathematical Surveys and Monographs, 202. American Mathematical Society, Providence, RI, 2015.
- [2] Y. Gordon, Some inequalities for Gaussian processes and applications. Israel J. Math. 50 (1985), 265-289.
- [3] V. D. Milman, New proof of the theorem of A. Dvoretzky on sections of convex bodies, Funct. Anal. Appl. 5(4) (1971), 288-295.
- [4] V. D. Milman, G. Schechtman, Asymptotic theory of finite-dimensional normed spaces. With an appendix by M. Gromov. Lecture Notes in Mathematics, 1200. Springer-Verlag, Berlin, 1986.

- [5] V. D. Milman and G. Schechtman, Global versus Local asymptotic theories of finite dimensional normed spaces, *Duke Math. Journal* 90 (1997), 73-93.
- [6] G. Schechtman, A remark concerning the dependence on ϵ in Dvoretzky's theorem. In: "Geometric aspects of functional analysis (1987-88)," *Lecture Notes in Math.*, 1376, pp. 274-277, Springer, Berlin (1989).